Matrix Population Models for Wildlife Conservation and Management

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Jean-Dominique LEBRETON, Jim HINES,
Xavier LAMBIN, Chris SUTHERLAND, Madan OLI

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Generation time

Alfred J. LOTKA,
The father of « mathematical demography »

Like most mathematicians, he takes the hopeful biologist to the edge of a pond, points out that a good swim will help his work, and then pushes him in and leaves him to drown.
Charles ELTON

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A classical age-dependent Leslie matrix

Pre birth-pulse matrix:
fecundities x 1st year survival = « net fecundities »

$$M = \begin{pmatrix} f_1 s_1 & f_2 s_1 & \ldots & f_i s_1 & \ldots & f_n s_n \\ s_2 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & s_3 & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & s_{i+1} & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & s_n & 0 \end{pmatrix}$$

Aging + survival:
survival probabilities
on 1st sub-diagonal

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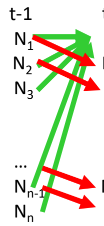
MANY age classes over ONE time step



$$N_1 = \Sigma f_i s_1 N_i(t-1)$$
$$N_2 = s_2 N_1(t-1)$$
$$N_3 = s_3 N_2(t-1)$$
$$N_{n-1} = s_{n-1} N_{n-2}(t-1)$$
$$N_n = s_n N_{n-1}(t-1)$$

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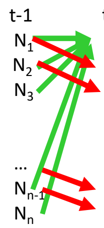
MANY age classes over ONE time step



$$N_1 = \Sigma f_i s_1 N_i(t-1)$$
expands as

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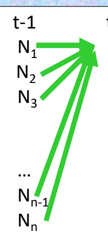
MANY age classes over ONE time step



$$N_1 = f_1 s_1 N_1(t-1) + f_2 s_1 N_2(t-1) + f_3 s_1 N_3(t-1) + \dots + f_{n-1} s_1 N_{n-1}(t-1) + f_n s_1 N_n(t-1)$$

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ONE age class over MANY time steps

t-n ... t-3 t-2 t-1 t

N_1 ... N_1 → N_1, N_2 → N_1, N_2, N_3, ..., N_{n-1}, N_n

$$N_1 = f_1\, s_1\, N_1(t-1) + f_2\, s_1\, s_2\, N_1(t-2) + f_3\, s_1\, s_2\, s_3\, N_1(t-3) + \dots + f_{n-1} s_1 s_2 \dots s_{n-1} N_1(t-n+1) + f_n\; s_1\, s_2 \dots s_n\; N_1(t-n)$$

Renewal equation $N_1(t) = \Sigma\, f_i\, l_i\, N_1(t-i)$ with $l_i = s_1 \dots s_i$

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Euler-Lotka equation

$N_1(t) = \Sigma\, f_i\, l_i\, N_1(t-i)$ with $l_i = s_1 \dots s_i$

However, under asymptotic regime: $N_1(t-i) = \lambda^{-i}\, N_1(t)$

Hence: $N_1(t) = \Sigma\, f_i\, l_i\, \lambda^{-i}\, N_1(t)$

i.e. $1 = \Sigma\, f_i\, l_i\, \lambda^{-i}$

Valid for any number of age classes (even ∞)

Euler - Lotka equation

1760 and 1911, respectively

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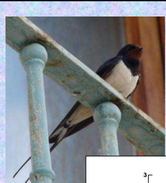
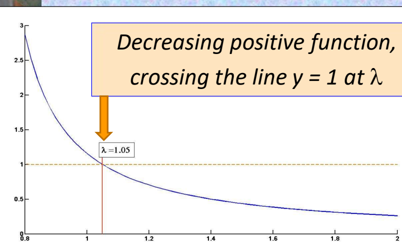
Euler-Lotka equation

Swallow example



Decreasing positive function, crossing the line y = 1 at λ



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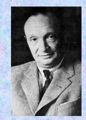
Two sides of the same coin



Stable Pop.Theory
Euler - Lotka equation
1760 and 1911, respectively

↔

Leslie Matrices
Age-structured Matrix models
1945, 1948

Stable Population Theory
to be expanded later
to stage-structured matrix models

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Generation time

time	t-n	…	t-3	t-2	t-1	t
contribution	$f_n\, l_n\, \lambda^{-n}$	…	$f_3\, l_3\, \lambda^{-3}$	$f_2\, l_2\, \lambda^{-2}$	$f_1\, l_1\, \lambda^{-1}$	$\Sigma = 1$
age of mothers at t	n	…	3	2	1	

Stable distribution of the Age of mothers at birth

Mean age of mothers at birth (in asymptotic regime)

$$T = \Sigma\, i\, f_i\, l_i\, \lambda^{-i}$$

$\bar{T}$, *Generation time, Leslie 1966,*
Hereafter denoted as T

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Derived quantities

Number of individuals per mother in next generation

$$R_0 = \Sigma\, f_i\, l_i$$

Mean age of child birth along a mother's life

$$T_C = \Sigma\, i\, f_i\, l_i \,/\, \Sigma\, f_i\, l_i$$

Cohort Generation Time

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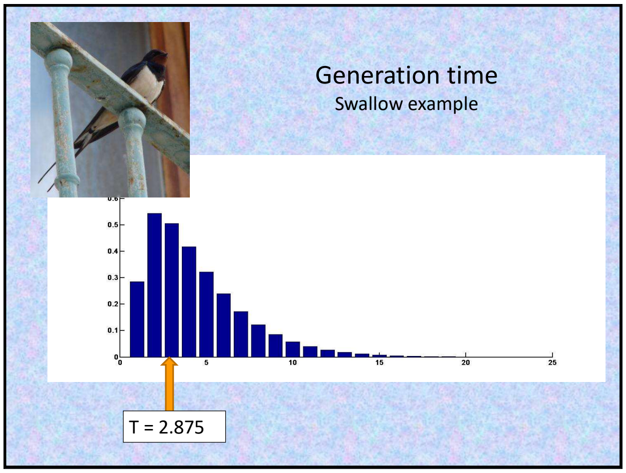
Generation time
Swallow example
0
5
10
15
20
25
T = 2.875

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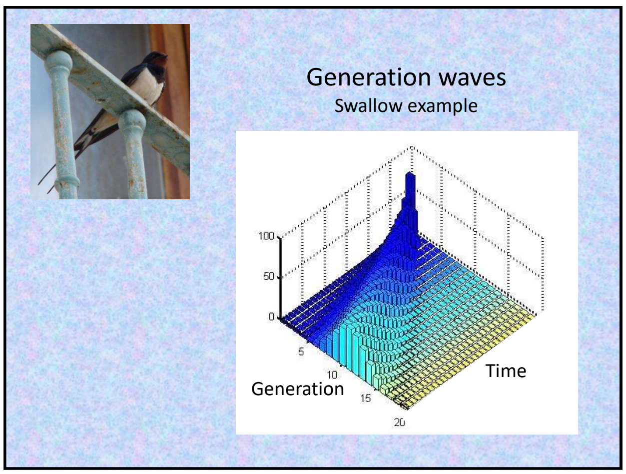
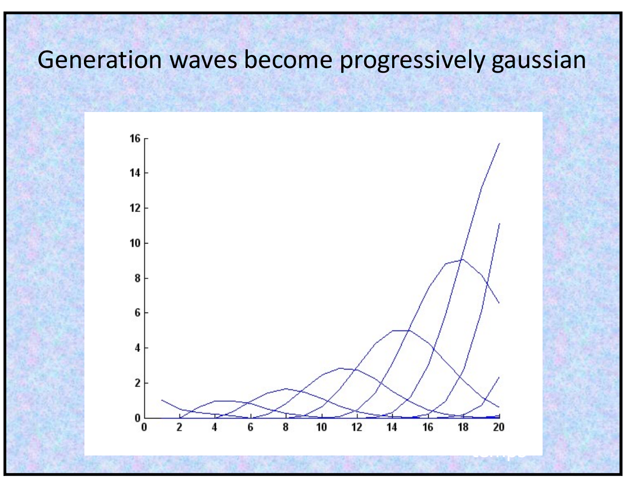

Generation waves
Swallow example
100
50
0
5
10
Generation
15
20
Time

14


Generation waves become progressively gaussian
16
14
12
10
8
6
4
2
0
0 2 4 6 8 10 12 14 16 18 20

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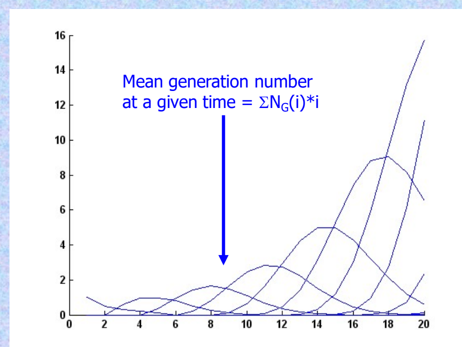
Generation waves become progressively gaussian



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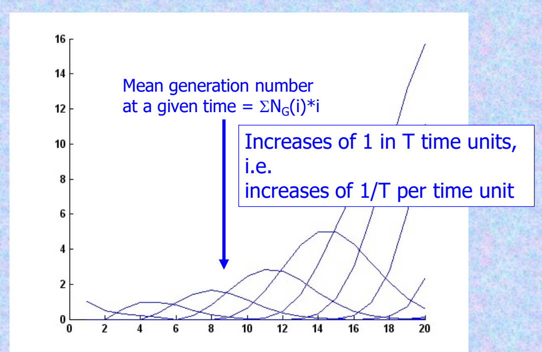
Generation waves become progressively gaussian



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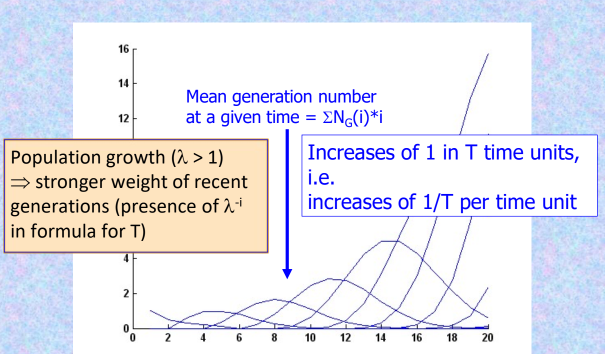
Generation waves become progressively gaussian



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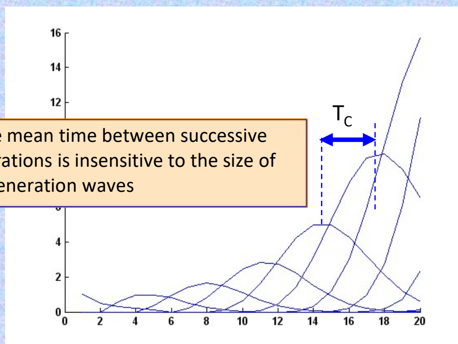
Generation waves become progressively gaussian



While mean time between successive generations is insensitive to the size of the generation waves

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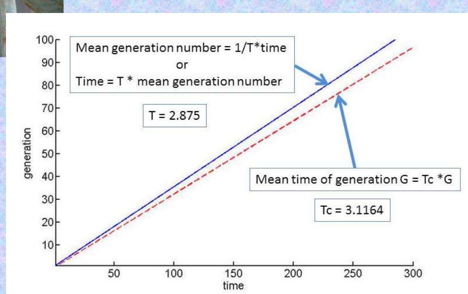
Generation waves
E(time/G) vs E(G/time)
Swallow example



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Nathan Keyfitz
1913-2010

Generation time and Sensitivity Analysis

The Euler-Lotka equation is an implicit function, linking any generic parameter θ and λ:

$$\phi(\theta,\lambda) = \Sigma\, f_i(\theta) l_i(\theta)\lambda(\theta)^{-i} = 1$$

If $\theta \rightarrow \theta + d\theta$, $\lambda \rightarrow \lambda + d\lambda$, but ϕ remains equal to1, i.e. $d\phi = 0$

As a consequence $$0 = \frac{\partial\phi}{\partial\lambda} d\lambda + \frac{\partial\phi}{\partial\theta} d\theta$$

Hence $$d\lambda/d\theta = -(\partial\phi/\partial\theta) / (\partial\phi/\partial\lambda)$$

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Generation time and Sensitivity Analysis

$d\lambda/d\theta = -(\partial\phi/\partial\theta) / (\partial\phi/\partial\lambda)$

From $\phi = \Sigma f_i l_i \lambda^{-i}$, $\partial\phi/\partial\lambda = \Sigma i f_i l_i \lambda^{-i-1} = -T / \lambda$

From $l_i = s_1 \ldots s_i$, $\partial\phi/\partial s' = \Sigma f_i (l_i / s_1)\lambda^{-i} = (1/s_1) \Sigma f_i l_i \lambda^{-i} = 1/s_1$

Sensitivity $d\lambda/ds_1 = \lambda/(s_1 T)$

Elasticity $(s_1/\lambda) d\lambda/ds_1 = 1/T$

same result for a change in all fecundities
same result for all parameters < 1st reprod.
(« immature parameters »)

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Generation time and turnover

Elasticity $(s_1/\lambda) d\lambda/ds_1 = 1/T$

However $(s_1/\lambda) d\lambda/ds_1 = u_1 v_1$
under $\Sigma u_i v_i = 1$

Hence $1/T = u_1 v_1$ under $\Sigma u_i v_i = 1$

A measure of turnover:

- *the proportion of new individuals, in reproductive value*
- *Also the asymptotic increase in mean generation number per year*

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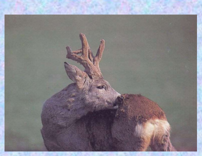

Sensitivity analysis

Survival

$M \rightarrow M_h = (1-h)M$ $MV=\lambda V \Rightarrow (1-h)MV=(1-h)\lambda V$

Hence $M_h V = (1-h)\lambda V$

$\lambda \rightarrow \lambda_h = (1-h)\lambda$, asymptotic structure V unchanged

x % change in all $s_i \rightarrow$ x % change in λ

The elasticity of λ wrt to $\{ s_1, s_2, \ldots, s_i, \ldots\}$ is 1

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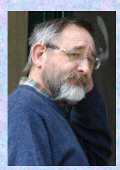
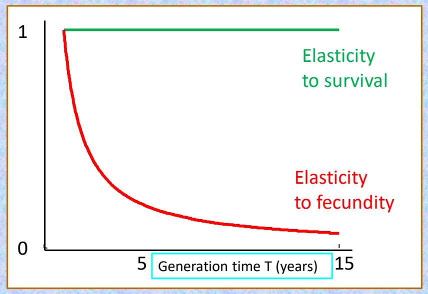


Jean Clobert

Sensitivity analysis

Fecundity and Survival
Lebreton and Clobert 1991



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Sensitivity analysis and Generation time

Albatross, T≈24 y:
-30 % in Fecundity ⇒
-1.25 % in growth rate

In any sharp-decline of a long lived species, first suspect a change in survival

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